

Distribution of Avalanche Sizes in the Hysteretic Response of the Random-Field Ising Model on a Bethe Lattice at Zero Temperature

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We consider the zero-temperature single-spin-flip dynamics of the random-field Ising model on a Bethe lattice in the presence of an external field h . We derive the exact self-consistent equations to determine the distribution $Prob(s)$ of avalanche sizes s as the external field increases from $-\infty$ to ∞ . We solve these equations explicitly for a rectangular distribution of the random fields for a linear chain and the Bethe lattice of coordination number $z = 3$, and show that in these cases, $Prob(s)$ decreases exponentially with s for large s for all h on the hysteresis loop. We find that for $z \geq 4$ and for small disorder, the magnetization shows a first-order discontinuity for several continuous and unimodal distributions of the random fields. The avalanche distribution $Prob(s)$ varies as $s^{-3/2}$ for large s near the discontinuity.

KEY WORDS: Random-field Ising model; hysteresis; Barkhausen noise; avalanches.

I. INTRODUCTION

Analytical treatment of problems having quenched disorder is usually difficult. There are few models having nontrivial quenched disorder that can be solved exactly. In this paper, we obtain exact results for the non-equilibrium properties of the random-field Ising model (RFIM) on the Bethe lattice. We consider the single-spin-flip Glauber dynamics of the system at zero temperature, as the external magnetic field is slowly varied from $-\infty$ to $+\infty$. As the field increases, the magnetization increases as groups of

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spins flip up together. This model has been proposed as a model of the Barkhausen noise by Sethna *et al.*⁽¹⁾ (see also ref. 2). In this paper, we set up the exact self-consistent equations satisfied by the generating function of the distribution of avalanche sizes, and analyze these to determine the behavior of the avalanche distribution function on the Bethe lattice.

The study of the equilibrium properties of the RFIM has been an important problem in statistical physics for a long time. In 1975, Imry and Ma,⁽³⁾ showed that arbitrarily weak disorder destroys long-ranged ferromagnetic order in dimensions $d < 2$. The persistence of ferromagnetism in $d = 2$ was a matter of a long controversy, but has now been established.⁽⁴⁾ A recent review of earlier work on this model may be found in ref. 5. As far as an exact calculation of thermodynamic quantities is concerned, there are only a few results. For example, Bruinsma studied the RFIM on a Bethe lattice in the absence of an external field and for a bivariate random-field distribution.⁽⁶⁾ There are no known exact results for the average free energy or magnetization, for a continuous distribution of random field, even at zero temperature and in zero applied field.

The non-equilibrium properties of the RFIM has attracted a lot of interest lately, arising from the observation by Sethna *et al.*⁽¹⁾ that its zero-temperature dynamics provides a simple model for the Barkhausen noise and return point memory. Barkhausen noise is the high frequency noise generated due to the small jumps in magnetization observed when ferromagnets are placed in oscillating magnetic fields (Fig. 1). Understanding and reduction of this noise is important for the design of many electronic devices.⁽⁷⁾ Experimentally it is observed⁽⁸⁻¹¹⁾ that the the increase of the magnetization occurs in bursts that span over two decades of size and the distribution of burst (avalanche) sizes seems to follow a power law in this range. Similar avalanche-like relaxational events are also observed in

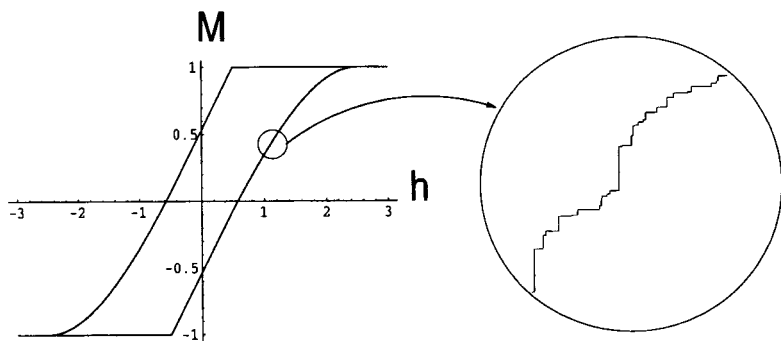


Fig. 1. The Hysteresis loop of magnetization M versus the external field h for RFIM. The zoomed figure shows the small jumps in magnetization that give rise to the Barkhausen noise.

other systems, for example, the stress-induced martensitic growth in some alloys.⁽¹²⁾ This power-law tail in the event-size distribution was interpreted by Cote and Meisel⁽¹¹⁾ as an example of *self-organized criticality*. But Percović *et al.*⁽²⁾ have argued that large bursts are exponentially rare, and the approximate power-law tail of the observed distribution comes from crossover effects due to nearness of a critical point. Recently Tadić⁽¹³⁾ has presented some evidence from numerical simulations that the exponents for avalanche distribution can vary continuously with disorder. Our results about the behavior of the avalanche distribution function also relate to this question whether any fine-tuning of parameters is required to see power-law tails in the avalanche distribution in the RFIM, and if the exponents can be varied continuously with disorder.

The advantage of working on the Bethe lattice is that the usual BBGKY hierarchy of equations for correlation functions closes, and one can hope to set up exact self-consistent equations for the correlation functions. The fact that Bethe's self-consistent approximation becomes exact on the Bethe lattice is useful as it ensures that the approximation will not violate any general theorems, e.g., the convexity of thermodynamic functions, sum rules. In the presence of disorder, in spite of the closure of the BBGKY hierarchy, the Bethe approximation is still very difficult, as the self-consistent equations become functional equations for the probability distribution of the effective field. These are not easy to solve, and available analytical results in this direction are mostly restricted to one dimension,⁽¹⁴⁾ or to models with infinite-ranged interactions.⁽¹⁵⁾ On the Bethe lattice, for short-ranged interactions with quenched disorder, e.g., in the prototypical case of the $\pm J$ random-exchange Ising model, the average free energy is trivially determined in the high temperature phase, but not in the low-temperature phase. It has not been possible so far to determine even the ground-state energy exactly despite several attempts.⁽¹⁶⁾

Calculation of time-dependent or non-equilibrium properties presents its own difficulties, even in the absence of disorder. Usually, for $d > 1$, one has to resort to the limit of coordination number becoming large, with interaction strength scaled suitably with coordination number to give a non-trivial thermodynamic limit.⁽¹⁷⁾ The large- d limit in the self-consistent field approximation for quantum-mechanical problems is similar in spirit.⁽¹⁸⁾

The RFIM model on a Bethe lattice is special in that the zero-temperature *nonequilibrium* response to a slowly varying magnetic field can be determined exactly.⁽¹⁹⁾ To be precise, the average non-equilibrium magnetization in this model can be determined exactly if the magnetic field is increased very slowly, from $-\infty$ to $+\infty$, in the limit of zero temperature. It thus provides a good theoretical model to study the slow relaxation to equilibrium in glassy systems. The dynamics is governed by the existence of

many metastable states, with large energy barriers separating different metastable states. We hope that this study of non-equilibrium response in this model would help in the more general problem of understanding the statistical mechanics of metastable states in glassy systems.

A brief summary of our results is as follows. We derive the exact self consistent equations for the generating function of the avalanche size distribution function $Q(x)$ on the Bethe lattice. This is a polynomial equation in $Q(x)$ and x , in which the coefficients depend on the external field h , and the distribution of the quenched random fields. We can solve these equations explicitly numerically and thus determine the qualitative behavior of the distribution of avalanches for any distribution of the quenched random fields. The behavior depends on the coordination number z , and on the details of the distribution function. We work out the distribution of avalanches explicitly for a rectangular distribution of the quenched fields, for the linear chain ($z=2$), and the 3-coordinated Bethe lattice. In both cases, one finds only exponential decay. We also studied other unimodal continuous distributions, e.g., when the random-field distribution is gaussian, or of the form $Prob(h_i) = (1/2\Delta) \text{sech}^2(h_i/\Delta)$, also for large z . We find that, for $z \geq 4$, there is a regime of disorder strengths for which the magnetization shows a jump-discontinuity (“first-order transition”), but the avalanche distribution, averaged over the hysteresis loop also shows a power law tail of the form $s^{-5/2}$ (“critical fluctuations”).

The paper is organized as follows. In Section II, we define the model precisely. In Section III, we briefly recapitulate the derivation of self-consistent equations for the magnetization in our model, and then use a similar argument to construct the generating function for the avalanche distribution for arbitrary distribution of the quenched random field. We set up a self-consistent equation for the probability Q_n , that an avalanche propagating in subtree flips exactly n more spins in the subtree before stopping. The probability distribution of avalanches is expressed in terms of this generating function. In Section IV, we consider the special case of a rectangular distribution of the random field. In this case, we explicitly solve the self-consistent equations for Bethe lattices with coordination numbers $z=2$ and 3. However, this case is non-generic. For small strength of disorder Δ , the magnetization jumps from -1 to $+1$ at some value of the field, but for larger disorder, when the system shows finite avalanches, there is no jump in magnetization and the distribution function decays exponentially for large s . In Section V, we analyse the self-consistent equations to determine the form of the avalanche distribution for some other unimodal continuous distributions of the random field. We find that in each case for coordination number $z \geq 4$, the magnetization shows a first-order jump discontinuity as a function of the applied field at some field-strength h_{disc} , for weak

disorder. Just below $h = h_{\text{disc}}$, the avalanche distribution has a universal $(-3/2)$ power-law tail. Section VI contains a discussion of our results, and some concluding remarks. Some algebraic details of the analytical solution for the rectangular distribution of quenched fields are relegated to two appendices.

II. DEFINITION OF THE MODEL

We consider a uniform Cayley tree of n generations where each non-boundary site has a coordination number z (see Fig. 2). The first generation consists of a single vertex. The r -th generation has $z(z-1)^{r-2}$ vertices for $r \geq 2$.

The RFIM on this graph is defined as follows: At each vertex there is a Ising spin $s_i = \pm 1$ which interacts with nearest neighbors through a ferromagnetic interaction J . There are quenched random fields h_i at each site i drawn independently from a continuous distribution $p(h_i)$. The entire system is placed in an externally applied uniform field h . The Hamiltonian of the system is

$$H = -J \sum_{\langle i, j \rangle} s_i s_j - \sum_i h_i s_i - h \sum_i s_i \quad (1)$$

We consider the response of this system when the external field h is slowly increased from $-\infty$ to $+\infty$. We assume the dynamics to be zero-temperature single-spin-flip Glauber dynamics, i.e., a spin flip is allowed only if the process lowers energy. We assume that if the spin-flip is allowed, it occurs with a rate Γ , which is much larger than the rate at which the

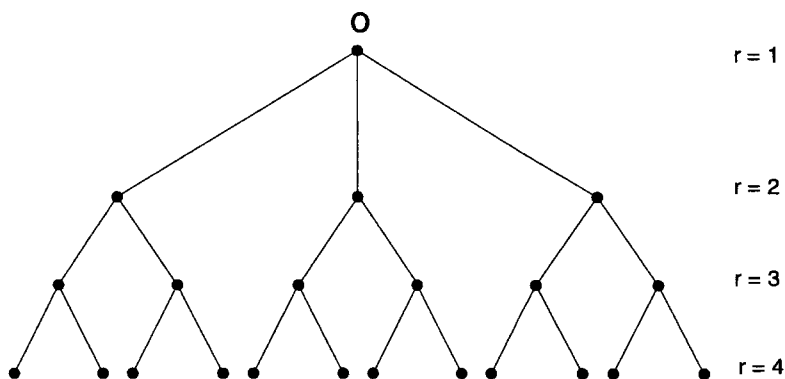


Fig. 2. A Cayley tree of coordination number 3 and 4 generations.

magnetic field h is increased. Thus we assume that all flippable spins relax instantly, so that the spin s_i is always parallel to the net local field ℓ_i at the site:

$$s_i = \text{sign}(\ell_i) = \text{sign} \left(J \sum_{j=1}^z s_j + h_i + h \right) \quad (2)$$

We start with $h = -\infty$, when all spins are down and slowly increase h . As we increase h , some sites where the quenched random field is large positive will find the net local field positive, and will flip up. Flipping a spin makes the local field at neighboring sites increase, and in turn may cause them to flip. Thus, the spins flip in clusters of variable sizes. If increasing h by a very small amount causes s spins to flip up together, we shall call this event an avalanche of size s . As the applied field increases, more and more spins flip up until eventually all spins are up, and further increase in h has no effect.

III. THE SELF-CONSISTENT EQUATIONS

The special property of the ferromagnetic RFIM that makes the analytical treatment possible is this: Suppose we start with $h = -\infty$, and all spins down at $t = 0$. Now we change the field slowly with time, in such a way that $h(t) \leq h(T)$, for all times $t < T$. Then the configuration of spins at the final instant $t = T$ does not depend on the detailed time dependence of $h(t)$, and is the same for all histories, so long as the condition $h(t) \leq h(T)$ for all earlier times is obeyed. In particular, if the maximum value $h(T)$ of the field was reached at an earlier time t_1 , then the configuration at time T is exactly the same as that at time t_1 . This property is called the return point memory.⁽¹⁾ We may choose to increase the field suddenly from $-\infty$ to $h(T)$ in a single step. Then, once the field becomes $h = h(T)$, several spins would have positive local fields. Suppose there are two or more such flippable sites. Then flipping any one of them up can only increase the local field at other unstable sites, as all couplings are ferromagnetic. Thus to reach a stable configuration, all such spins have to be flipped, and *the final stable configuration reached is the same, and independent of the order in which various spins are relaxed.* This property will be called the abelian property of relaxation. Using the symmetry between up and down spins, it is easy to see that the abelian property also holds whether the new value of field h'' is greater or less than its initial value h' so long as one considers transition from a stable configuration at h' to a stable configuration at h'' .

We first briefly recapitulate the argument of our earlier paper⁽¹⁹⁾ which uses the abelian nature of spin-flips to determine the mean magnetization for any field h in the lower half of the hysteresis loop by setting up a self-consistent equation.

Since the spins can be relaxed in any order, we relax them in this: first all the spins at generation n (the leaf nodes) are relaxed. Then spins at generation $n-1$ are examined, and if any has a positive local field, it is flipped. Then we examine the spins at generation $n-2$, and so on. If any spin is flipped, its descendant are reexamined for possible flips.⁽²⁰⁾ In this process, clearly the flippings of different spins of the same generation r are independent events.

Suppose we pick a site at random in the tree away from the boundary, the probability that the local field at this site is positive, given that exactly m of its neighbors are up, is precisely the probability that the local field h_i at this site exceeds $[(z-2m)J-h]$. We denote this probability by $p_m(h)$. Clearly,

$$p_m(h) = \int_{(z-2m)J-h}^{\infty} p(h_i) dh_i \quad (3)$$

Let $P^{(r)}(h)$ be the probability that a spin on the $(n-r)$ th generation will be flipped when its parent spin at generation $n-r-1$ is kept down, the external field is h , and each of its descendent spins has been relaxed. As each of the $z-1$ direct descendents of a spin is independently up with probability $P^{(r-1)}$, it is straightforward to write down a recursion relation for $P^{(r)}$ in terms of $P^{(r-1)}$. For $r \gg 1$, these probabilities tend to limiting value P^* , which satisfies the equation⁽¹⁹⁾

$$P^*(h) = \sum_{m=0}^{z-1} \binom{z-1}{m} [P^*(h)]^m [1 - P^*(h)]^{z-1-m} p_m(h) \quad (4)$$

For the spin at O , there are z downward neighbors, and the probability that it is up is given by

$$Prob(s_O = +1 | h) = \sum_{m=0}^z \binom{z}{m} [P^*(h)]^m [1 - P^*(h)]^{z-m} p_m(h) \quad (5)$$

Because all spins deep inside the tree are equivalent, $Prob(s_O = +1 | h)$ determines the average magnetization for all sites deep inside the tree. Using Eqs. (4) and (5), we can determine the magnetization for any value of the external field h . This determines the lower half of the hysteresis loop. The upper half is obtained similarly.

Now consider the state of the system at external field h , and all the flippable sites have been flipped. We increase the field by a small amount dh till one more site becomes unstable. We would like to calculate the probability that this would cause an “avalanche” of n spin flips. Since all sites deep inside are equivalent, we may assume the new susceptible site is the site O .

It is easy to see that this avalanche propagation is somewhat like propagation of infection in the contact process on the Bethe lattice. The “infection” travels *downwards* from the site O which acts as the initiator of infection. If any site is infected, then it can cause infection of some of its descendents. If the descendent spin is already up, it cannot be flipped; such sites act as immune sites for the infection process. If the descendent spin is down, it can catch infection with a finite probability. Furthermore, this probability does not depend on whether the other “sibling” sites catch infection. Infection of two or more descendents of an infected site are uncorrelated events. Thus, we can expect to find the distribution of avalanches on the Bethe lattice, as for the size distribution of percolation clusters on a Bethe lattice.⁽²¹⁾ However, a precise description in terms of the contact process is complicated, as here the infection spreads in a correlated background of “immune” (already up) spins, and the probability that a site catches infection does depend on the number of its neighbors that are already up.

We start with the initial configuration of all spins down. Now increase the external field to the value h . Consider a site X at some generation $r > 1$ of the Cayley tree (Fig. 3). We call the subtree formed by X and its descendents T_X , the subtree rooted at X . We keep its parent spin Y at generation $r - 1$ down, and relax all the sites in T_X at the uniform field h . If X is far away from the boundary, the probability that spin at X is up is $P^*(h)$. The conditional probability that spin at a descendent of X is up, given that the spin at X is down is also $P^*(h)$. We measure the response of T_X to external perturbation by forcibly flipping the spin at Y (whatever the local field there) and see how many spins in this subtree flip in response to this perturbation. Let Q_n be the probability that the spin at X was down when Y was down and n spins on the subtree T_X flip up if S_Y is flipped up. Here allowed values of n are 0, 1, 2, ... Clearly, we have

$$P^* + \sum_{n=0}^{\infty} Q_n = 1 \quad (6)$$

We define

$$Q(x) = \sum_{n=0}^{\infty} Q_n x^n \quad (7)$$

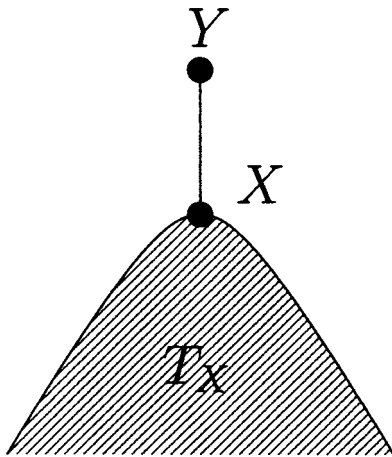


Fig. 3. A sub-tree T_X formed by X and its descendants. The sub-tree is rooted at X and Y is the parent spin of X .

Clearly, $Q(x=0) = Q_0$ and $Q(x=1) = 1 - P^*$. It is straight forward to write the self-consistent equation for $Q(x)$. Let us first relax all spins on T_X keeping X and Y down. The probability that exactly m the descendants of X are turned up in this process be denoted by $Pr(m)$. Clearly

$$Pr(m) = \binom{z-1}{m} P^{*m} (1 - P^*)^{z-1-m} \quad (8)$$

For a given m , the conditional probability that local field at X is such that spin remains down, even if Y is turned up is $1 - p_{m+1}$. Summing over m , and using the expression for $Pr(m)$ above, we get

$$Q_0 = \sum_{m=0}^{z-1} \binom{z-1}{m} P^{*m} (1 - P^*)^{z-1-m} [1 - p_{m+1}] \quad (9)$$

We can write down an expression for Q_1 similarly. In this case, if m of the direct descendants of X are up when Y is down, the local field at all the remaining $z-1-m$ direct descendants must be such that they remain down even if X is flipped up. This probability is $\binom{z-1}{m} P^{*m} Q_0^{z-1-m}$. The local quenched field at X must satisfy $(z-2m)J - h > h_X > (z-2m-2)J - h$. The probability for this to occur is $p_{m+1} - p_m$. Hence we get

$$Q_1 = \sum_{m=0}^{z-1} (p_{m+1} - p_m) \binom{z-1}{m} P^{*m} Q_0^{z-1-m} \quad (10)$$

The equation determines Q_n for higher n can be written down similarly. It only involves the probabilities Q_m with $m < n$ for the descendent spins. These recursion equations are expressed more simply in terms of the generating function $Q(x)$. It is easily checked that the self-consistent equation for $Q(x)$ is

$$Q(x) = Q(x=0) + x \sum_{m=0}^{z-1} \binom{z-1}{m} (p_{m+1} - p_m) P^{*m} Q(x)^{z-1-m} \quad (11)$$

This is a polynomial equation in $Q(x)$ of degree $z-1$, whose coefficients are functions of h through $P^*(h)$ and $p_m(h)$. It is easily checked that for $x=1$, the ansatz $Q(x=1) = 1 - P^*$ satisfies the equation, as it should. To determine $Q(x)$ for any given external field h , we have to first solve the self-consistent equation for P^* [Eq. (4)]. This then determines $Q(x=0)$ using Eq. (9), and then, given P^* and $Q(0)$, we solve for $Q(x)$ by solving the $(z-1)$ -th degree polynomial equation Eq. (11).

Finally, we express the relative frequency of avalanches of various sizes when the external field is increased from h to $h + dh$ in terms of $Q(x)$. Let $G_s(h) dh$ be the probability that avalanche of size s is initiated at O . We also define the generating function $G(x | h)$ as

$$G(x | h) = \sum_{s=1}^{\infty} G_s(h) x^s \quad (12)$$

Consider first the calculation of $G_s(h)$ for $s=1$. Let the number of descendents of O that are up at field h be m . For the spin at site O to be down at h , but flip up at $h + dh$, the local field h_O must satisfy $[(z-2m)J - (h + dh)] < h_O < [(z-2m)J - h]$. This occurs with probability $p(zJ - 2mJ - h) dh$. Each of the $(z-m)$ down neighbors of O must not flip up, even when s_O flips up. The conditional probability of this event is Q_0^{z-m} . Multiplying by the probability that m neighbors are up, we finally get

$$G_1(h) = \sum_{m=0}^z \binom{z}{m} P^{*m} Q_0^{z-m} p(zJ - 2mJ - h) \quad (13)$$

Arguing similarly, we can write the equation for $G_s(h)$ for $s=2, 3$ etc. These equations simplify considerably when expressed in terms of the generating function $G(x | h)$, and we get

$$G(x | h) = x \sum_{m=0}^z \binom{z}{m} P^{*m} Q(x)^{z-m} p(zJ - 2mJ - h) \quad (14)$$

In numerical simulations, and experiments, it is much easier to measure the avalanche distribution integrated over the full hysteresis loop. To get the probability that an avalanche of size s will be initiated at any given site O in the interval when the external field is increased from h_1 to h_2 , we just have to integrate $G(x | h)$ in this range. For any h , the value of dG/dx at $x = 1$ is proportional to the mean size of an avalanche, and thus to the average slope of the hysteresis loop at that h .

IV. EXPLICIT CALCULATION FOR THE RECTANGULAR DISTRIBUTION

While the general formalism described in the previous section can be used for any distribution, and any coordination number, to calculate the avalanche distributions explicitly, we have to choose some specific form for the probability distribution function. In this section, we shall consider the specific choice of a rectangular distribution: The quenched random field is uniformly distributed between $-\Delta$ and Δ , so that

$$p(h_i) = \frac{1}{2\Delta}, \quad \text{for } -\Delta \leq h_i \leq \Delta \quad (15)$$

In this case, the cumulative probabilities $p_m(h)$ become piecewise linear functions of h , and h -dependence of the distribution is easier to work out explicitly. We shall work out the distributions for the linear chain ($z = 2$), and the 3-coordinated Bethe lattice.

A. The Linear Chain

The simplest illustration is for a linear chain. In this case the self-consistent equation, for the probability P^* [Eq. (4)] becomes a linear equation. This is easily solved, and explicit expressions for Q_0 , and $Q(x)$ are obtained (see Appendix A). The different regimes showing different qualitative behavior of the hysteresis loops are shown in Fig. 4.

For $h < 2J - \Delta$ (region A), all the spins remain down. For $h > \Delta$, all spins are up (region D). For $\Delta < J$, we get a rectangular loop and the magnetization jumps discontinuously from -1 to $+1$ in a single infinite avalanche, and we directly go from region A to D as the field is increased. For $\Delta > J$, we get nontrivial hysteresis loops.

The hysteresis loops for different values of $\Delta = 0.5, 1.5$ and 2.5 are shown in Fig. 5. If Δ is sufficiently large ($\Delta > J$), we find that the mean magnetization is a precisely linear function of the external field for a range

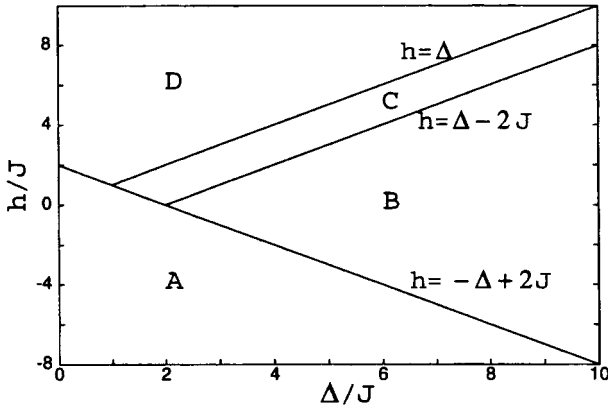


Fig. 4. Behavior of RFIM in the magnetic field - disorder ($h - \Delta$) plane for a linear chain. The regions A-D correspond to qualitatively different responses. In region A all spins are down and in region D all are up. The avalanches of finite size occur in region B and C.

of values of the external field h (region B in Fig. 2). For larger h values, the magnetization shows saturation effects, and is no longer linear (region C).

The explicit forms of the generating function $Q(x)$ are given in the Appendix A. We find that in region B, the function $Q(x)$ is independent of the applied field h . The distribution function $G_s(h)$ has a simple dependence on s of the form

$$G_s(h) = A_1 s \left(\frac{J}{\Delta} \right)^s \quad (16)$$

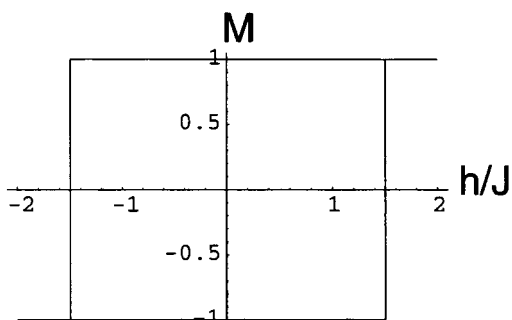
where A_1 is a constant, that depends only on J/Δ , and does not depend on s or h

$$A_1 = \frac{1}{2\Delta} \frac{(1 - J/\Delta)^2}{(J/\Delta)} \quad (17)$$

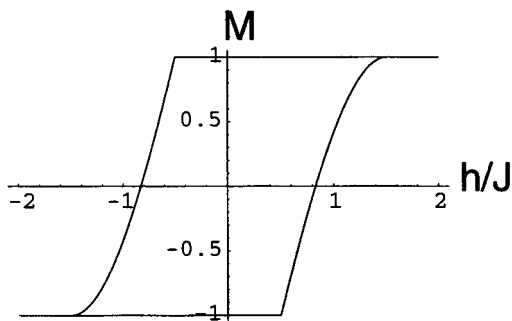
In region C, the mean magnetization is a nonlinear function of h . But $Q(x)$ is still a rational function of x . From the explicit functional form of $Q(x)$ and $G(x | h)$ are given in the Appendix A, we find that $G_s(h)$ is of the form

$$G_s(h) = [A'_1 s + A'_2] \left(\frac{J}{\Delta} \right)^s, \quad \text{for } s \geq 2 \quad (18)$$

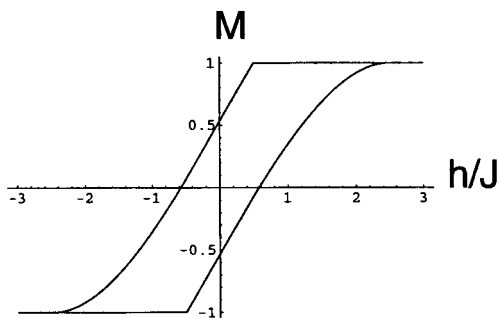
Here A'_1 and A'_2 have no dependence on s but are explicit functions of h .



(a)



(b)



(c)

Fig. 5. Hysteresis loops for the linear chain for the rectangular distribution of quenched fields with different widths (a) $\Delta/J = 0.5$, (b) $\Delta/J = 1.5$ and (c) $\Delta/J = 2.5$.

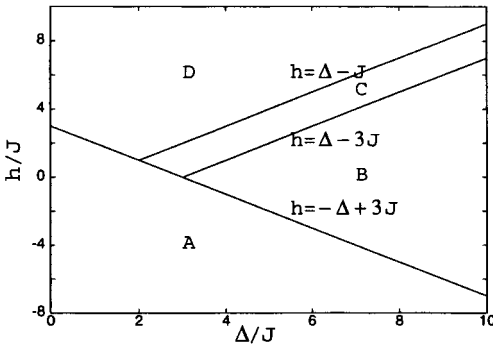


Fig. 6. Behavior of RFIM in the magnetic field-disorder ($h-\Delta$) plane for Bethe lattice of coordination number 3. The qualitative behavior in different regions A–D is similar to that of a linear chain (Fig. 4).

Integrating over h from $-\infty$ to ∞ we get the integrated avalanche distribution D_s ,

$$D_s = \int_{-\infty}^{\infty} G_s(h) dh \quad (19)$$

It is easy to see from above that the integrated distribution D_s also has the form

$$D_s = [A_2 s + B_2] \left(\frac{J}{\Delta}\right)^s, \quad \text{for } s \geq 2 \quad (20)$$

where the explicit forms of the coefficients A_2 and B_2 are given in the Appendix A.

B. The Case $z=3$

The analysis for the case $z=3$ is very similar to the linear case. In this case, the self-consistent equation for $P^*(h)$ [Eq. (4)] becomes a quadratic equation. The qualitative behavior of solution is very similar to the earlier case. Some details are given in Appendix B. We again get regions A–D as before, but the boundaries are shifted a bit, and are shown in Fig. 6. As before, in region B, the average magnetization is a linear function of h , and the avalanche distribution is independent of h .

We find that in regime B, the distribution of avalanche sizes is given by

$$G_s(h) = N \left[\frac{(2s)!}{(s-1)!(s+2)!} \right] (1 - J/\Delta)^s \left(\frac{J}{\Delta}\right)^s \quad (21)$$

where N is a normalization constant given by

$$N = \frac{3}{2A} (1 - J/A)^2 \frac{1}{(J/A)} \quad (22)$$

It is easy to see that for large s , $G_s(h)$ varies as

$$G_s \sim s^{-3/2} \kappa^s \quad (23)$$

where

$$\kappa = 4(1 - J/A)(J/A) \quad (24)$$

In region B, J/A is always less than $1/3$, and so this function always has an exponential decay for large s .

In the region C, we find that the avalanche distribution is of the form

$$G_s(h) = N' \left[\frac{(2s)!}{(s-1)! (s+2)!} \right] \kappa^s \quad (25)$$

where N' is a normalization constant independent of s , and κ is a cubic polynomial in the external field h :

$$\begin{aligned} \kappa = & \frac{1}{8(1 - 2J/A)^2} [\{9 - 53(J/A) + 119(J/A)^2 - 107(J/A)^3\} \\ & + \{ -5 + 10(J/A) + 11(J/A)^2 \} (h/A) \\ & + \{ 3 - 9(J/A)^2 \} (h/A)^2 + (h/A)^3] \end{aligned} \quad (26)$$

As κ is not a very simple function of h , explicit expressions for the integrated distribution D_s are hard to write down.

V. GENERAL DISTRIBUTIONS

The analysis of the previous section can, in principle, be extended to higher coordination numbers, and other distributions of random fields. However, the self-consistent equations become cubic, or higher order polynomials. In principle, an explicit solution is possible for $z \leq 5$, but it is not very instructive. However, the qualitative behavior of solutions is easy to determine, and is the same for all $z \geq 4$. We shall take $z = 4$ in the following for simplicity. Since we only study the general features of the self-consistent equations, we need not pick a specific form for the continuous distributions

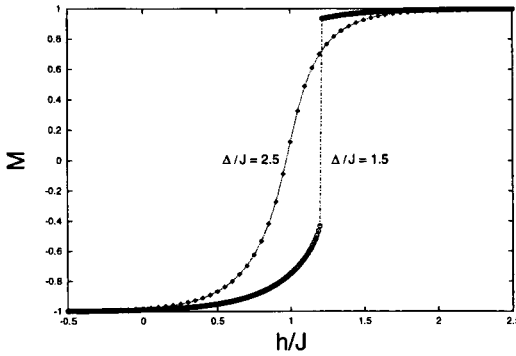


Fig. 7. Magnetization as a function of increasing field for the Bethe lattice with $z = 4$ and the random field distribution given by Eq. (28).

of random field distribution $p(h_i)$. We shall only assume that it has a single maximum around zero and asymptotically go to zero at $\pm\infty$.

For small width (Δ) of the random field distribution, i.e., for weak disorder the magnetization shows a jump discontinuity as a function of the external uniform field, which disappears for a larger values of Δ .⁽¹⁹⁾ For fields h just lower than the value where the jump discontinuity occurs, the slope of the hysteresis curves is large, and tends to infinity as the field tends to the value at which the jump occurs. This indicates that large avalanches are more likely just before the first order jump in magnetization (Fig. 7).

For $z = 4$, the self-consistent equation for $P^*(h)$ [Eq. (4)] is cubic

$$aP^{*3} + bP^{*2} + cP^* + d = 0 \quad (27)$$

where a, b, c and d are functions of the external field h , expressible in terms of the cumulative probabilities $p_i, i = 0$ to 3 ,

$$a = p_3 - 3p_2 + 3p_1 - p_0$$

$$b = 3p_2 - 6p_1 + 3p_0$$

$$c = 3p_1 - 3p_0 - 1$$

$$d = p_0$$

This equation will have 1 or 3 real roots, which will vary with h . We have shown this variation for the real roots which lie between 0 and 1 in Fig. 8 for the case where $p(h_i)$ is a simple distribution

$$p(h_i) = \frac{1}{2\Delta} \operatorname{sech}^2(h_i/\Delta) \quad (29)$$

We have also solved numerically the self-consistent equation for P^* for other choices of $p(h_i)$, like the gaussian distribution, and for higher z ($=4, 5, 6$). In each case we find that the qualitative behavior of the solution is very similar. Note that the rectangular distribution discussed in the previous section is very atypical in that both the coefficients a and b vanish for an entire range of values of h .

In the generic case, we find two qualitatively different behaviors: For larger values of Δ , there is only one real root for any h . For Δ sufficiently small, we find a range of h where there are 3 real solutions. There is a critical value Δ_c of the width which separates these two behaviors. For the particular distribution chosen $\Delta_c \simeq 2.10382$.

In the first case, the real root is a continuous function of h , and correspondingly, the magnetization is a continuous function of h . This is the case corresponding to $\Delta = 2.5$ in Fig. 8.

For smaller $\Delta < \Delta_c$, for large $\pm h$ there is only one root, but in the intermediate region there are three roots. The typical variation is shown for $\Delta = 1.5$ in Fig. 8. In the increasing field the probability $P^*(h)$ initially takes the smallest root. As h increases, at a value $h = h_{\text{disc}}$, the middle and the lower roots become equal and after that both disappear from the real plane. At $h = h_{\text{disc}}$ the probability $P^*(h)$ jumps to the upper root. Thus for $\Delta < \Delta_c$ there is a discontinuity in $P^*(h)$ which gives rise to a first order jump in the magnetization curve.

The field h_{disc} where the discontinuity of magnetization occurs, is determined by the condition that for this value of h , the cubic equation [Eq. (27)] has two equal roots. The value of P^* at this point, denoted by P^*_{disc} , satisfies the equation

$$3a_0 P_{\text{disc}}^{*2} + 2b_0 P_{\text{disc}}^* + c_0 = 0 \quad (29)$$

where a_0, b_0 and c_0 are the values of a, b and c at $h = h_{\text{disc}}$.

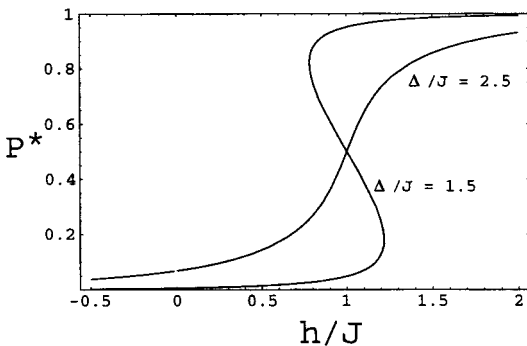


Fig. 8. Variation of $P^*(h)$ with h for the Bethe lattice with $z=4$, and the random field distribution given by Eq. (28).

We now determine the behavior of the avalanche generating function $G_s(h)$ for large s and h near h_{disc} . The behavior for large s corresponds to x near 1. So we write $x = 1 - \delta$, with δ small, and $h = h_{\text{disc}} - \varepsilon$. Near h_{disc} , a, b, \dots vary linearly with ε and

$$P^* \approx P_{\text{disc}}^* - \alpha \sqrt{\varepsilon} + O(\varepsilon) \quad (30)$$

where α is a numerical constant.

Since $Q(x=1) = 1 - P^*(h)$, if x differs slightly from unity $Q(x)$ also differs from $1 - P^*(h)$ by a small amount. Substituting $x = 1 - \delta$ and $Q(x = 1 - \delta) = 1 - P^* - F(\varepsilon, \delta)$ in the self-consistent equation for $Q(x)$ [Eq. (11)], where both δ and F are small, using Eq. (29), we get to lowest order in δ, ε and F

$$F^2 + \beta \sqrt{\varepsilon} F - \gamma^2 \delta = 0 \quad (31)$$

where β and γ are some constants. Thus, to lowest orders in ε and δ , F is given by

$$F = (1/2) [\sqrt{\beta^2 \varepsilon + 4\gamma^2 \delta} - \beta \sqrt{\varepsilon}] \quad (32)$$

Thus $Q(x)$ has leading square root singularity at $x = 1 + (\beta^2 \varepsilon / 4\gamma^2)$. Consequently, $G(x|h)$ will also show a square root singularity $x = 1 + (\beta^2 \varepsilon / 4\gamma^2)$. This implies that the Taylor expansion coefficients $G_s(h)$ vary as

$$G_s(h) \sim s^{-3/2} \left(1 + \frac{\beta^2 \varepsilon}{4\gamma^2}\right)^{-s}, \quad \text{for large } s \quad (33)$$

At $\varepsilon = 0$, we get

$$G_s(h_{\text{disc}}) \sim s^{-3/2} \quad (34)$$

Thus at $h = h_{\text{disc}}$ the avalanche distribution has a power law tail.

To calculate the integrated distribution D_s , we have to integrate Eq. (33) over a range of ε values. For large s , only $\varepsilon < \gamma^2 / \beta^2 s$ contributes significantly to the integral, and thus we get

$$D_s \sim s^{-5/2}, \quad \text{for large } s \quad (35)$$

Thus the integrated distribution shows a robust $(-5/2)$ power law for a range of disorder strength A .

VI. DISCUSSION

In this paper, we set up exact self-consistent equations for the avalanche distribution function for the RFIM on a Bethe lattice. We were able to solve these equations explicitly for the rectangular distribution of the quenched field, for the linear chain $z=2$, and the 3-coordinated Bethe lattice. For more general coordination numbers, and general continuous distributions of random fields, we argued that for very large disorder, the avalanche distribution is exponentially damped, but for small disorder, generically, one gets a jump in magnetization, accompanied by a square-root singularity. For field-strengths just below corresponding to the jump discontinuity, the avalanche distribution function has a power-law tail of the form $s^{-3/2}$. The integrated avalanche distribution then varies as $s^{-5/2}$ for large s .

Some unexpected features of the solution deserve mention. Firstly, we find that the behavior of the self-consistent equations for $z=3$ is qualitatively different from that for $z>3$. The behavior for the linear chain ($z=2$) is, of course, expected to be different from higher z . One usually finds same behavior for all $z>2$. Mathematically, the reason for this unusual dependence is that the mechanism of two real solutions of the polynomial equation merging, and both becoming unphysical (complex) is not available for $z=3$. Here the self-consistency equation is a quadratic, and from physical arguments, at least one of the roots must be real. That a Bethe lattice may show non-generic behavior for low coordination numbers has been noted earlier by Ananikyan *et al.* in their study of the Blume–Emery–Griffiths model on a Bethe lattice. These authors observed that the qualitative behavior for $z<6$ is different from that for $z\geq 6$.⁽²²⁾

The second point we want to emphasize is that here we find that the power-law tail in the distribution function is accompanied by the first-order jump in magnetization. Usually, one thinks of critical behavior and first-order transitions as mutually exclusive, as first-order jump pre-empts a build-up of long-ranged correlations, and all correlations remain finite-ranged across a first-order transition. This is clearly not the case here. In fact, the power-law tail in the avalanche distribution disappears, when the jump disappears. A similar situation occurs in equilibrium statistical mechanics in the case of a Heisenberg ferromagnet below the critical temperature. As the external field h is varied across zero, the magnetization shows a jump discontinuity, but in addition has a cusp singularity for small fields.⁽²³⁾ But in this case the power-law tail is seen on *both sides of the transition*.

Note that for most values of disorder, and the external field, the avalanche distribution is exponentially damped. We get robust power law

tails in the distribution, only if we integrate the distribution over the hysteresis cycle across the magnetization jump. But, in this case, the control parameter h is swept across a range of values, in particular across a (non-equilibrium) phase transition point! In this sense, while no explicit fine-tuning is involved in an experimental setup, this is not a self-organized critical system in the usual sense of the word. Recently Pázmándi *et al.* have argued that the hysteretic response of the Sherrington–Kirkpatrick model to external fields at zero temperature shows self-organized criticality for all values of the field.⁽²⁴⁾ However, this seems to be because of the presence of infinite-ranged interactions in that model.

The treatment of this paper may be extended to the site-dilution case discussed by Tadić.⁽¹³⁾ From the structural stability of the mechanism which leads to the cusp singularity just before the jump-discontinuity in magnetization, it is clear that in our model, introduction of site dilution would not change the qualitative behavior of solutions.

A general question concerns the behavior of the avalanches for more general probability distributions. Clearly, if $p(h_i)$ has a discrete part, it would give rise to jumps in p_i as a function of h , and hence give rise to several jumps in the hysteresis loop. These could preempt the cusp singularity mechanism which is responsible for the power-law tails. If the distribution $p(h_i)$ is continuous, but multimodal, then it is possible to have more than one first order jump in the magnetization.⁽²⁵⁾ This is confirmed by explicit calculation in some simple cases. If $p(h_i)$ has power-law singularities, these would also lead to power-law singularities in p_i , and hence in $P^*(h)$. Even for purely continuous distributions, the merging of two roots as the magnetic field varies need not always occur. For example, it is easy to check that for the rectangular distribution, even for $z \geq 4$, we do not get a power law tail for any value of Δ . The precise conditions necessary for the occurrence of the power-law tail are not yet clear to us.

Finally, we would like to mention some open questions. Our analysis relied heavily on the fact that initial state was all spins down. Of course, we can start with other initial conditions. It would be interesting to set up self-consistent field equations for them. In particular, the behavior the return loop, when the external field is increased from $-\infty$ to some value h_1 , and then decreased to a lower value h_2 seems an interesting quantity to determine. Another extension would be to make the rate of field-sweep comparable to the single-spin flip rate (still assuming $T=0$ dynamics). This would mean some large avalanches in different parts of the sample could be evolving simultaneously. Then one could study the sweep-rate dependence of the hysteresis loops, and the frequency dependence of the Barkhausen noise spectra. This is perhaps of some relevance in real experimental data,

and would also make contact with other treatments of Barkhausen noise that focus on the domain wall motion.

APPENDIX A

For the case of a linear chain, the self-consistent equation, for the probability P^* [Eq. (4)] is a linear equation, whose solution is,

$$P^*(h) = \frac{p_0}{[1 - (p_1 - p_0)]} \quad (\text{A1})$$

For $h < 2J - \Delta$, p_0 is zero, and hence $P^*(h)$ is zero, and all the spin remain down (region A in Fig. 4).

For $h > 2J - \Delta$, and $\Delta < J$, p_1 is 1 whenever p_0 is nonzero. Then from Eq. (A1), $P^*(h)$ becomes 1. Thus, for $\Delta < J$, we get a rectangular loop and the system changes from all spins down to all spins up state in a single big avalanche.

For $\Delta > J$, $p_1 - p_0$ equals J/Δ and is independent of h , in the range $2J - \Delta < h < \Delta$. Thus $P^*(h)$ is a linear function of h in this range, increasing from 0 to 1.

Defining

$$\varepsilon = \frac{1}{2} \left(1 + \frac{h}{\Delta} - \frac{2J}{\Delta} \right) \quad (\text{A2})$$

we obtain the expression for P^* as

$$P^*(h) = \begin{cases} 0 & \text{for } \varepsilon < 0 \\ \frac{\varepsilon}{1 - J/\Delta} & \text{for } 0 \leq \varepsilon \leq 1 - J/\Delta \\ 1 & \text{for } \varepsilon > 1 - J/\Delta \end{cases} \quad (\text{A3})$$

Using Eq. (9), the expression for Q_0 is,

$$Q_0 = (1 - p_1) - (p_2 - p_1) P^*(h) \quad (\text{A4})$$

The generating function $Q(x)$ obtained from the self-consistent equation [Eq. (11)] is,

$$Q(x) = \frac{Q_0 + xP^*(p_2 - p_1)}{1 - x(p_1 - p_0)} \quad (\text{A5})$$

and the generating function $G(x | h)$ given by Eq. (14) becomes,

$$G(x | h) = x \{ [Q(x)]^2 p(2J - h) + 2P^* [Q(x)] p(-h) + P^{*2} p(-2J - h) \} \quad (\text{A6})$$

Now if $\Delta > 2J$, and $-\Delta + 2J < h < \Delta - 2J$ (region B in Fig. 4),

$$(p_2 - p_1) = (p_1 - p_0) = J/\Delta,$$

$$p(2J - mJ - h) = \frac{1}{2\Delta} \quad \text{for } m = 0, 1, 2$$

and

$$P^* + Q_0 = 1 - J/\Delta$$

Thus

$$Q(x) = \frac{1 - (J/\Delta)}{1 - (J/\Delta)x} - P^* \quad (\text{A7})$$

and

$$G(x | h) = \frac{x}{2\Delta} [P^* + Q(x)]^2 = \frac{x}{2\Delta} \frac{(1 - J/\Delta)^2}{(1 - xJ/\Delta)^2} \quad (\text{A8})$$

Expanding $G(x | h)$ in powers of x , we get the probability distribution of avalanches in region B given by Eq. (16) of Section IV.A.

In the region C, p_2 saturates to value 1, $p(-2J - h)$ becomes zero and $(p_2 - p_1)$ becomes $(1 - J/\Delta - \varepsilon)$. Thus we get,

$$Q_0 = \frac{(1 - J/\Delta - \varepsilon)^2}{(1 - J/\Delta)} \quad (\text{A9})$$

In terms of P^* and Q_0 we get

$$Q(x) = \frac{Q_0 + xP^*[1 - 2(J/\Delta) - \varepsilon]}{1 - (J/\Delta)x} \quad (\text{A10})$$

and

$$G(x | h) = \frac{x}{2\Delta} \{ [P^* + Q(x)]^2 - P^{*2} \} \quad (\text{A11})$$

Expanding $G(x | h)$ in powers of x we get, in region C

$$G_1(h) = \frac{1}{2\Delta} [(P^* + Q_0)^2 - P^{*2}] \quad (\text{A12})$$

and

$$G_s(h) = [A'_1 s + A'_2] \left(\frac{J}{\Delta}\right)^s, \quad \text{for } s \geq 2 \quad (\text{A13})$$

Here A_2 and B_2 have no dependence on s but are explicit functions of h

$$A'_1 = \frac{1}{2\Delta} \left[\frac{1}{(J/\Delta)} (P^* + Q_0)^2 + \frac{1}{(J/\Delta)^2} (P^* + Q_0) P^* \left(1 - \frac{2J}{\Delta} - h/\Delta\right) + \frac{1}{4(J/\Delta)^3} P^{*2} \left(1 - \frac{2J}{\Delta} - h/\Delta\right)^2 \right]$$

$$A'_2 = \frac{-1}{\Delta} \left[\frac{1}{2(J/\Delta)^2} (P^* + Q_0) P^* \left(1 - \frac{2J}{\Delta} - h/\Delta\right) + \frac{1}{4(J/\Delta)^3} P^{*2} \left(1 - \frac{2J}{\Delta} - h/\Delta\right)^2 \right]$$

Integrating over h from $-\infty$ to ∞ we get the integrated avalanche distribution D_s ,

$$D_s = \int_{-\infty}^{\infty} G_s(h) dh \quad (\text{A14})$$

where

$$D_1 = \frac{1}{(1 - J/\Delta)^2} \left[1 - 6 \left(\frac{J}{\Delta}\right) + 14 \left(\frac{J}{\Delta}\right)^2 - \frac{46}{3} \left(\frac{J}{\Delta}\right)^3 + \frac{47}{6} \left(\frac{J}{\Delta}\right)^4 - \frac{9}{5} \left(\frac{J}{\Delta}\right)^5 \right] \quad (\text{A15})$$

and, for $s \geq 2$,

$$D_s = (A_2 s + B_2) \left(\frac{J}{\Delta}\right)^s \quad (\text{A16})$$

with

$$A_2 = \frac{1}{30(J/\Delta)} \left[30 - 110 \left(\frac{J}{\Delta} \right) + 135 \left(\frac{J}{\Delta} \right)^2 - 54 \left(\frac{J}{\Delta} \right)^3 \right]$$

$$B_2 = \frac{1}{15(1-J/\Delta)} \left[5 - 10 \left(\frac{J}{\Delta} \right) + 4 \left(\frac{J}{\Delta} \right)^2 \right]$$

APPENDIX B

For $z = 3$, the self-consistent equation, for $P^*(h)$ [Eq. (4)] is a quadratic equation,

$$[(p_2 - p_1) - (p_1 - p_0)] P^*(h)^2 + [2(p_1 - p_0) - 1] P^*(h) + p_0 = 0 \quad (\text{B1})$$

For the rectangular distribution, the coefficient of P^{*2} is zero for a range of h -values, and $P^*(h)$ is still a piecewise linear function of h

$$P^*(h) = \begin{cases} 0 & \text{for } \varepsilon < 0 \\ \frac{\varepsilon}{1 - 2(J/\Delta)} & \text{for } 0 \leq \varepsilon \leq 1 - 2(J/\Delta) \\ 1 & \text{for } \varepsilon > 1 - 2(J/\Delta) \end{cases} \quad (\text{B2})$$

where ε is defined as,

$$\varepsilon = \frac{1}{2} \left(1 + \frac{h}{\Delta} - \frac{3J}{\Delta} \right) \quad (\text{B3})$$

The self-consistent equation for $Q(x)$ [Eq. (11)] becomes,

$$x(p_1 - p_0)[Q(x)]^2 + [2xP^*(p_2 - p_1) - 1] Q(x) + xP^{*2}(p_3 - p_2) + Q_0 = 0 \quad (\text{B4})$$

where Q_0 is obtained [Eq. (9)] as

$$Q_0 = (1 - p_1) - 2(p_2 - p_1) P^* + [(p_2 - p_1) - (p_3 - p_2)] P^{*2} \quad (\text{B5})$$

and the expression (14) for $G(x | h)$ becomes,

$$G(x | h) = x \{ [(Q(x))]^3 p(3J - h) + 3[Q(x)]^2 P^* p(J - h) + 3[Q(x)] P^{*2} p(-J - h) + P^{*3} p(-3J - h) \} \quad (\text{B6})$$

Now in the region B,

$$(p_3 - p_2) = (p_2 - p_1) = (p_1 - p_0) = J/\Delta,$$

$$p(3J - 2mJ - h) = \frac{1}{2\Delta} \quad \text{for } m = 0 \text{ to } 3$$

and

$$P^* + Q_o = 1 - J/\Delta$$

Solving Eq. (B4) and choosing the root which is well behaved for x near 0, we get

$$Q(x) = \frac{1 - \sqrt{1 - 4(J/\Delta) x(P^* + Q_o)}}{2(J/\Delta) x} - P^* \quad (\text{B7})$$

and the expression for the integrated distribution (B6) becomes

$$G(x | h) = \frac{x}{2\Delta} [P^* + Q(x)]^3 \quad (\text{B8})$$

Expanding $G(x)$ in power series of x , we obtain the Eq. (21) of Section IV.B.

In the region C, p_3 saturates to the value 1, $p(-3J - h)$ becomes zero and $(p_3 - p_2)$ is no longer independent of h . Substituting the appropriate expressions, we find that

$$Q(x) = \frac{1 - \sqrt{1 - 4(J/\Delta) x[(1 - 3(J/\Delta) - \varepsilon) + (P^* + Q_o)]}}{2(J/\Delta) x} - P^* \quad (\text{B9})$$

and

$$G(x | h) = \frac{x}{2\Delta} \{ [P^* + Q(x)]^3 - P^{*3} \} \quad (\text{B10})$$

We note that the term inside the radical sign in $Q(x)$, and also in $G(x | h)$, is a simple linear function of x . It is thus straightforward to expand it in powers of x using binomial expansion. This gives us the Eq. (25) of Section IV.B.

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20. This step is not really necessary if we are only interested in determining the magnetization at the site O . Skipping this step leads to considerable simplification of the relaxation process: first the spins of generation n are examined, then those of $(n - 1)$ etc. till we finally examine the spin at O . No spin is checked more than once. The resulting configuration is not fully relaxed, but it is easy to prove that further relaxation will not change the state of the spin at O . The argument can be extended to show that the probability that an avalanches starting at O is of size s also is the same in this partially relaxed state as in the fully relaxed state.
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23. Below T_c , the magnetization goes as

$$m \sim \text{sign}(h)[m_0 + A|h|^{(d-2)/2}], \quad \text{as } h \rightarrow 0$$

See G. Parisi, *Statistical Field Theory* (Addison-Wesley, 1988), p. 195.

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25. This would happen if $P^*(h)$ as a function of h shows a “double s” curve. Then there must be at least 4 values of h for which the slope of the curve is infinite. This is possible only if the equation determining P_{disc}^* (variant of Eq. (29)) is at least a quartic, hence only if $z \geq 6$.